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FRACTAL PARAMETERS OF SLUM SETTLEMENTS: WAVE-SPECTRUM ANALYSIS (STUDY OF KARACHI)

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Abstract

Fractal parameters of slum settlements, such as those in Karachi, can provide valuable insights into these communities' spatial organization and growth patterns. In this research, considering the slum areas or Kachi Abadies (KAs) as an 'urban malignancy,' the spectral (spectrum) analysis method and the Fourier transform are used to analyze the patterns of population distribution in Karachi city based on its spatial dynamics. The study uses mathematical experiments and empirical analysis to calculate the fractal dimension for exploring urban fractal parameter relationships form and growth. The population data and land use data of Karachi Kachi Abadis (KKAs) are used to attain the numerical values of all parametric relations. The calculated results showed a close numerical coherence.

Keywords: Fourier Transformation, Spectral Analysis, Malignancy, Fractal Dimension, Density

Introduction

Anderson discussed that cities have all four major and classical characteristics (i.e., first, fast unrestrained growth, incursion, and devastation of neighboring ecosystems or space, distant colonization (metastasis), and fourth, de-differentiation) of a malignant process (Anderson, 1961). It has been observed since the early 19th century that urban growth is rapidly increasing in developed and developing countries. This explosive urban growth, particularly in developing countries, has raised many problematic hurdles and barriers. Because of this rapid increase in urban growth, unorganized and non-saturated settlements can be seen in the city centers and peripheral areas, badly affecting the city's spatial patterns (shape). Therefore, we have termed these unorganized settlements (slums) or Kachi Abadies (KA) as "malignant areas." The conventional measures like length, area, and density based on Euclidean geometry do not affect urban form and growth because of their scale-free property. Euclidean geometry gives the dimensions in whole numbers, like the topological dimension of a point is 0-D, for a line 1-D, and for a surface it exhibits 2-D. However, many matters do not fit these categories. Great examples are snowflakes, the coastline of Britain, and cities. The dimensions of these bodies are somewhere between 1 and 2. Thus, it possesses a fractional dimension that justifies the fractal dimension of irregular shape. Similarly, cities exhibit fractional dimensions (i.e., $1 \leq D \geq 2$). Mathematical forms like fractals can examine the shape of developing cities. The self-similarity property of fractals may be regarded as the mapping of cities; we may call these shapes "fractal patterns." The fractal dimension value for big cities is close to 2, and the fractal dimension value may decrease or increase according to the city size. Thanks to Benoit Mandelbrot

(Mandelbrot, 1999), who introduced this new form of mathematics in 1983, which enabled us to describe the chaotic and irregular pattern of bodies, and this mathematical form was termed "Fractal Geometry." This geometry helped researchers of all fields, particularly demographers and mathematicians, to explore the new world and provided them with an essential tool for studying irregular shapes, urban form, and growth. The pioneers who introduced fractals in urban studies are (1985), Batty and Longley (Batty, Longley, 1987), Frankhauer and Sadler (Frankhauser, Sadler, 1991), (Benguigui, Daoud, 1991), (Fotheringham et al., 1989; White, Englelen, 1993), etc. With rapid and uncontrolled growth in urbanization and the form of fractal cities, various neoteric aspects have evolved to analyze fractal cities.

Rationale of Study

Studying fractal parameters of slum settlements, such as those in Karachi, can provide valuable insights into these communities' spatial organization and growth patterns. Wave-spectrum analysis is a method used to analyze fractals, which can reveal patterns and characteristics of the settlement that may not be apparent through traditional analysis methods. This information can be useful for urban planners and policymakers in developing strategies for improving the living conditions and overall well-being of residents in these settlements. It can also be useful in understanding the growth of urbanization and its impact on the environment and society.

Significance of Study

The study of fractal parameters of slum settlements, such as those in Karachi, can have significant practical implications for several different fields.

- Urban Planning: The information obtained from the study can be used by urban planners to develop strategies for improving the living conditions and overall well-being of residents in slum settlements. It can also inform decisions about managing and regulating urban growth in these areas.

- Environmental Impact: The study of fractal parameters can also provide insights into the environmental impact of slum settlements, such as their effects on air and water quality and their potential to contribute to climate change.

- Socio-Economic Development: The study can also help policymakers understand the social and economic conditions in slum settlements and how these conditions may affect the city's overall development.

- Housing and infrastructure provision: The study can also provide information for the provision of housing and infrastructure for the residents.

- Public Health and Epidemiology: The study can also be useful in understanding the impacts of slum settlements and the spread of diseases and inform the development of public health policies and interventions.

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Mathematical Models and Fractal Dimension Relations

Clark's Law Urban Density Functions

"Fractal Geometry plays two roles. It is the geometry of deterministic chaos, and it can also describe the geometry of mountains, clouds, and galaxies." (Benoit Mandelbrot)

Mandelbrot (1983) introduced a new form of mathematics that enabled us to describe the chaotic and irregular pattern of bodies, and this mathematical form was termed "Fractal Geometry." This geometry helped researchers of all fields, particularly demographers and mathematicians, to explore the new world. It gave them a critical means for learning irregular shapes, urban form, and growth. Fractal geometry is nature's geometry, transforming nature into Mathematics.

Further research related urban growth and form to an abstract idea of size, scale, shape, and dimension (Batty, 2008; Longley et al., 1991). An urban fractal power law can be reduced to two components of an exponential law (Chen, Zhou, 2003), and the exponential function greatly binds the power law and entropy (Clark, 1951). Thus, the negative exponential and inverse power functions (Smeed, 1963) associated with fractal cities are essential for such studies. The negative exponential function is much more plausible in the empirical analysis of urban population density among all other functions (Wangand, Zhou, 1999; Feng, 2002). The negative exponential function is mainly employed to describe the city's population density, and the inverse power function is usually used to identify the city's land usage density (Clark, 1951; Batty, Longley, 1994; Frankhauser, 1994). Cities' population density can be modeled empirically by the following exponential function: Clark's model (Clark, 1951).

$$D_{(x)} = \text{K}\ e^{-\mu x} \qquad (2.1)$$

here, $D_{(x)}$ represents the population density when x is the distance from the city ccenter$x = 0$), the constant K of proportionality ual to the central dede densityubscriptnsity D_0 , and μ is the characteristic radius of the population distribution also known as density parameter. This model on urban density can avail new insights to capture the dynamics (kinetics) of urban morphology (division) from the view of population (Chen, 2012). Thus, exponential distribution with $\mu = \frac{1}{x_0}$ we can write the negative exponential model in terms of central density as

$$D_{(x)} = D_0\, e^{-x/}x_0 \qquad (2.2)$$

The urban density power function can be represented as

$$D_{(x)} = D_1\, x^{-(d-D_a)} \qquad (2.3)$$

Where $D_{(x)}$ and x represent the same meanings as in (2.2), D_1 represents the proportionality constant, d = 2 represents space embedding dimension, D_a shows radial dimension of KAs (Slum Areas). Equation (2.3) is the form of Smeed's model which was initially suggested as population density model in early 1960s (Smeed, 1963). Smeed's model provides the theoretical interpretation of the rule of allometric development on urban area and population (Clark, 1951).

The Wave-Spectrum Relation of Urban Density

Theoretically, in a broad sense, practically every fractal dimension can be thought of as a correlation dimension (Smeed, 1963). Parr (1985a, 1985b) determined that the negative exponential function is better enough for relating density in the urban area itself, whereas the inverse power function is much suitable to the urban-fringe and surroundings (Liu,1992). The parameters of these functions exhibit the spatial form of their concerning systems from which they are estimated. The negative exponential distributions favor mostly the structures of classical geometry (non-fractal structures) of urban population, but using the Fourier transformation it can be correlated with fractal structures. Continuous spectrum of generalized dimension may possess the zero order correlation dimensions D_a (dimension of one-point correlation) and the second order correlation dimension$'B'$ (point-point correlation dimension) (Frankhauser, 1994; Hern, 2008). The variability or diversity of urban models proposes irregularity or regularity breakings (aspects) of topographical systems that refers to the spatial complexity of urban schemes (system) and urban growth. Now let a density function, $f(x)$, which obeys the following scaling rule

$$f(\mu x) \propto \mu^{-\alpha} f(x) \qquad (2.4)$$

Where μ is scale element, α scaling exponent i.e. $(\alpha = d - D_a)$, x is distance variable. Now if we apply Fourier transform to (2.4) then

$$F(\mu n) = \mathfrak{f}\,[f(\mu x)] = \mu^{-(1-d)}\,\mathfrak{f}\,[f(x)] = \mu^{-(1-d)}\,\mathrm{F}(n) \qquad (2.5)$$

Where f is the Fourier transform, n is the wave number, $F(n)$ is the image function or the original function.

We may extract the wave-spectrum relation from (2.5)

$$\mathrm{S}(n) \propto n^{-2(1-\alpha)} \qquad (2.6)$$

Here $S(n) = |F(n)|^2$, spectral density of energy.

Comparison can be used to find the numerical relationship between the spectral exponent and the fractal dimension. Equation (2.3) is related to the wave-spectrum because it is a function of scaling symmetry i.e. $\alpha = (d - D_a)$, thus (2.6) yields (Liu, Liu, 1992)

$$S(n) \propto n^{-2(1-d+D_a)} = n^{-2(D_a-1)} = n^{-B} \qquad (2.7)$$

Thus,

$$B = 2(D_a - 1) \qquad (2.8)$$

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The relationship between (B) the point-point correlation dimension, and (D_a) the one-point correlation dimension is shown in equation (2.8). The fractal dimension of the self-similar form of cities is represented by the parameter D_a. Another fractal dimension can be derived from wave-spectrum relation by using the dimensional investigations which is known as self-affine record dimension (D_s) (Mandelbrot, 1999); (Barnsley et al., 1988). Each part or segment of a self-affine item is a scaled-down version of the whole object (Wangand, Zhou, 1999). If D_s be the affine dimnsion than, B =2.9)

H represents the Hurst exponent (Hurst, et al. 1965). The Hurst exponent is mhe "index of dependence" or "index of long-range dependence". The concept of H is is also derived from the rescaled range analysis method, also known as R⁄S analysis (Hurst et al., 1965; Chen, 2010).

If the value of H ranges between 0.5 and 1, then it specifies a long-term positive autocorrelation in a time series. If the value ranges from 0 – 0.5, then it indicates a time series in which the high and low values in adjacent pairs flip over time. Where $\mathrm{H} = 0.5$ denotes a series that is tncorrelated,. However, it is rthevalue for series in which the autocorrelations at tiny time lags might be positive or negative. However, ttheaabsolute valuesof the autocorrelations decay exponentially qo zero (Batty, Longley, 1987). That's in contradiction to the power law decline tcommonlyobserved for $1\text{one} > H > 0.5$ and $0.5 > H > 0$ cases.

As we have defined the parameters D_a and D_s earlier and now we generate a relation between these parameters using equations (2.8) and (2.9) as,

$$D_a = \frac{7-2D_s}{2} = \frac{7}{2} - D_s \qquad (2.10)$$

Because the spatial action of particles in genuine urban expansion is regarded to be typical of Fractional Brownian Motion (fBm) rather than ordinary random walk, the D_s of actual cities ranges from 1 to 2 for a debate of fBm (Feder, 1988; Mandelbrot, Van Ness, 1968). Grounded on fBm, the relation between Hurst exponent versus autocorrelation coefficient of the growth series can be shown as (Liu, Liu, 1992; Feder, 1988)

$$A_c = 2^{2H-1} - 1 \qquad (2.11)$$

Where A_c is the autocorrelation coefficient. The correlation of a variable with itself in space is measured by spatial autocorrelation. It can be favorable (positive) or unfavorable (negative). When similar values occur close together (clustered together in space), positive spatial autocorrelation occurs when dissimilar values occur close together (dispersed in space), negative spatial autocorrelation occurs (Batty, Longley, 1994). When H=0.5, autocorrelation coefficient is zero indicating random walk (Brownian motion) which means that Moran's coefficient I = 0 (no autocorrelation) and when H > 0.5, A_c>0, indicating positive spatial autocorrelation means that the Moran's coefficient I is close to +1. Finally, when H < 0.5, A_c< 0, indicating negative spatial correlation means the Moran's exponent I is close to -1. Moran's coefficient 'I' is based upon the first-order lag 2-

dimensional spatial autocorrelation (Griffith, 2003), while auto correlation coefficient 'A_c'is grounded on the multiple-lag 1- dimensional spatial autocorrelation (Chen, 2010).

Using the calculated values of D_a and D_S we have calculated the values for H, B and A_c using the mathematical expression (2.8), (2.9), and (2.11) respectively. The instances are shown in **Table 1**. In theory, the value of D_a ranges from 0 to 2 and empirically 1 to 2 as shown in Table 1. The Hurst exponent H varies between 0 to 1 and the D_S ranges from 1 to 2, and the A_c varies between -1 to +1.

The inverse power law is approximated by the spectral density based on the Fourier transform of the negative exponential function (Chen, 2008). The scaling reaction is depicted by the spectral density of the negative exponential distribution (NED) that is given below (Chen, Zhou, 2008; Liu, Liu, 1992)

$$S(n) \propto n^{-B} = n^{-2} \qquad (2.12)$$

Where theoretical value of B is equal to 2, indicating $D_S = 3/2$.

The dimension relation in equation (2.10) is a very key relation to solve the different and difficult problems on cities studies and population dynamics including the scaling exponent of allometric relation between population and urban area. The equation (2.2) is used to describe the urban population density. Since $B \to 2$ according to (2.12), and according to (2.9) we have $D_S \to 1.5$. Now if we put these values in equation (2.10) we get $D_a \to 7/2 - 1.5 = 2$. As a result, dimension of urban occurrences that justify the NED may be studied as $D_a \to d_E = 2$.

Mathematical Experiments based on Negative Exponential Function

Spatial correlation function is quite useful in the study of urban density which gives a correlation between the random variables and distance between two different temporal points in space, or time. The urban density negative exponential model is essentially a specific spatial correlation function (Clark, 1951). Exponential function can be used to attain a power-law relation between spectrum density and wave number. Considering the relation constructed in (Chen, 2008), we can write the density-density correlation functions in the form as:

$$C(x) = \int_{-\infty}^{\infty} D(x)D(r+x)dx = 2D_0^2 \int_0^{\infty} e^{-2x/x_0 - x/x_0}\, dx \qquad (3.1)$$

Where $D(x)$ shows the population density of cell at distance *r* from the city center, $D(r+x)$ shows the population density of another cell at distance x from R. Specified $r = 0$, it is measured that one cell turn into the city center, and the spatial correlation function collapse into an exponential function (Clark, 1951)

$$C(x) = D(0)D(x) = D_0^2 e^{-x/x_0} \qquad (3.2)$$

If the data are so normalized that the center of city bears density $D_0 = 1$, we have $\mathrm{Cx} = D(x)$, and thus (3.2) is alike (2.2). For this instance, Clark's law (Clark, 1951)

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is just a special density-density correlation function that specifies spatial correlating action between the location and center of city, having the separation x from the center. The distance parameter, x_0, is comparative to the spatial correlation length. A greater value of the characteristic range (radius) (x_0) means a stretched correlation distance.

It is important to remember that the energy spectrum and the autocorrelation function can be transformed to one another using the Fourier's cosine transform (Clark, 1951) such as:

$$S(n) = \int_{-\infty}^{\infty} \mathrm{C}(x)\, e^{-i2\pi nx}\, dx2 = \int_{0}^{\infty} \mathrm{C}(x) \cos 2\pi nx\ dx, \qquad (3.3)$$

Where $i = \sqrt{-1}$ denotes the imaginary (complex number) unit, n shows wave-number i.e. the wavelength's reciprocal, $S(n)$ represents the spectral density of energy in the relevant frequency range. The concept of energy spectrum comes from engineering mathematics. The product of a function's Fourier transform and its conjugate is parallel to the mathematical form of energy in physics (Zhu, 1991). The Fourier transform of (3.2) can be written in the form due to the symmetry of the correlation function,

$$F(n) = D_0^2 \int_{-\infty}^{\infty} e^{-x/x_0}\ e^{-i2\pi nx}\, dx = \frac{2x_0 D_0^2}{1+i2\pi n x_0} \qquad (3.3)$$

For the slum settlements of city $2x_0 D_0^2$ is large enough therefore we have

$$[\frac{1}{2x_0 D_0^2}]^2 \to 0 \qquad (3.4)$$

As a result, the energy spectral density can be calculated using the energy integral [1, 33] as follows:

$$\mathrm{S}(n) = |F(n)|^2 = \frac{(2x_0 D_0^2)^2}{1+D_0^2(2\pi n)^2} = \frac{1}{(\pi n)^2/D_0^4} \propto\ n^{-2} \qquad (3.5)$$

Because the length of the sample path L is usually limited, the wave spectrum density $W(n) = |G(n)|^2\ /\ L$ is always employed to replace the energy spectrum density S(n) in practice so (3.5) can thus be rewritten as (Liu, Liu, 1994)

$$W(n) \propto n^{-2} \qquad (3.6)$$

Equation (3.6) is an approximation based on perfect conditions that may be generalized to the scaling relation such as:

$$W(n) \propto n^{-B} \qquad (3.7)$$

Where B is known as "*spectral exponent*" which typically varies between 0 and 3. When value of B is close 1, (3.7) specifies that is named as $1/B$ noise (see, e.g., (Bak, 1996; Mandelbrot, 1999). Actually, the fractal dimension of urban population profiles is connected with the spectral exponent. Using equations (2.6), (2.7) and (2.12) we can generate a relation such as

$$W(n) = \frac{S(n)}{N} \alpha\, n^{-B} \qquad (3.7b)$$

A fractal structure can be discovered for a time series or spatial series if the relationship between spectral density and frequency or wave number follows the scaling law provided by (3.7). The relationship between B and D for $d_E = 1$ dimension variables has been demonstrated (Saupe, 1991; Voss, 1988) as follows:

$$D = d_E + \frac{3-B}{2} = \frac{5-B}{2} = 2 - H \qquad (3.8)$$

Where d_E denotes the dimension of Euclidean space. Consequently, $B = 5 - 2D$, here D gives the fractal dimension of urban population profiles ($d_E < D < d_E + 1$), and H refers to the Hurst exponent ($0 \le H \le 1$) (Clark, 1951). Furthermore, the fractional Brownian motion can be used to obtain the autocorrelation coefficients of the rate of change, as shown in (Feder, 1988):

$$\mathrm{A_c} = \frac{(-D(x-1)D(x+1))}{(D(x)^2)} = 2^{2H-1} - 1 \qquad (3.8b)$$

The Study deduce from the prestigious field of time series analysis that equation (3.8b) yields a special density-density correlation function. There are many time series analysis methods that can be used to analyses spatial series (Bloomfield, 2000) and demographic data, such as autocorrelation analysis, auto regression analysis, and spectrum analysis. If $\mathrm{D} = 1.5$ or $\mathrm{B} = 2$, respectively, and $\mathrm{H} = \frac{1}{2}$, Ac=0 is attained. In this case, the mthcells only interact with the $m \pm 1)th$ cells directly; they do not interact with the $(m \pm 2)$th or above cells. If D is less than 1.5 or B is larger than 2, A_c must be greater than zero and H must be greater than 0.5. If D is more than 1.5 or 2, we have $H < 0.5$, and thus A_c is zero. In this situation, the m^{th} cell will have a negative effect on the $(m \pm u)th$ cells.

The fractal dimension is $\mathrm{D} \approx 3/2 \approx 1.5$, and the Hurst exponent is $\mathrm{H} = 2 - \mathrm{D} \approx 0.5$, which yields the autocorrelation coefficient $\mathrm{Ac} \approx 0$. For the negative exponential function of urban population density, the expected result of spectral exponent is $\mathrm{B} \approx 2$, hence the fractal dimension is $\mathrm{D} \approx 3/2 \approx 1.5$. This implies that city systems are spatially local. In Physics, the concept of locality states that objects that are far apart cannot have a direct influence on one another (Einstein, 1948). In other words, an object's immediate surroundings are the only thing that influences it. The fact that the spatial autocorrelation coefficient $Ac \to 0$ indicates that a population cell is exclusively interested in its immediate neighbours.

Results and Discussions

Karachi as the Case study: Empirical Results

The city is both a living structure and an open system that is constantly in motion. A city is born, grows, heals from wounds (wars, natural disasters, etc.) and occasionally dies in part or entirely (Courtat, Gloaguen, Douady, 2011). Its growth is shaped by internal

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and external restrictions, with local geography acting as a shell that sculpts the overall shape (Frankhauser, Sadler, 1991). This can include examining physical structures at various scales, as well as patterns of mobility and land use, among other things. MS Excel and Mat-lab can easily be used to apply spectral analysis on real cities (Smeed, 1963). In this study of wave-spectrum relation in urban studies, we use the population and land use of Karachi KAs (slums) as shown in **Figure 1**. Data of these KAs has been taken from the survey records of Orangi Pilot Project (OPP) Karachi (Volume 1, 2 and 3).

Clark's law governs the population density distribution (PDD) of Karachi KAs, and it may be adapted to any model (2.2). A computation using ordinary least squares (OLSs) produces:

$$D(x) = 31097e^{-x/}2.91$$

$R^2 = 0.998$ is a rough estimate of goodness of fit. For the radial fractal dimension (D_a) of Karachi KAs, the population within a specific radius $P(r)$, does not satisfy the power law.

Therefore population distribution of Karachi KAs cannot be depicted using the D_a , but it can be illustrated by the D_s. Thus the city's human activities may be based on Brownian motion (BM) and contain a collection of self-affine fractal records. The power law governs the relationship between wave number and spectral density so that:

$$W(n) = 6633.124\, n^{-2.10}$$

The estimated value of spectral exponent (B) is calculated ‘2.10’, which is too near to the theoretical expected value of ‘B’ that is 2.00. Now using (2.9) the profile dimension D_s can be estimated as,

$$D_s \approx \frac{5 - 2.084}{2} \approx 1.458$$

The result is almost equal to the predictable value of D_s which is 1.5. This result exhibits that the population distribution of Karachi slums possesses nearly nature of random walk (Smeed, 1963). Then, using (2.10), the city form’s D_a can be approximated as,

$$D_a = \frac{2.084}{2} + 1 \approx 2.042$$

This value of D_a is quite close to the Euclidean Dimension's theoretical value $D_a = d = 2$.

Two types of distributions (functions) are mainly important for this kind of study. The first one is the Negative Exponential Distribution (NED) that is largely applied to imitate a city’s population density and the second one is the Inverse Power Distribution (IPD) which is applied to characterize the urban land use density (Smeed, 1963). The land use density of city Karachi appears to convene the NED rather than the power-law distribution due to the underdevelopment of fractal structure. In the 2-dimensional space, (2.3) can be represented in the integral form as

$$N(x) = N_1 x^{D_a} \qquad (3.9)$$

Where N(x) gives the pixel number showing the land use area within a radius of x from the city centre and N_1 is a constant. Using the data of urban used in (3.9) gives

$$N(x) = 3.146\, x^{2.428} \qquad (3.10)$$

The D_a of urban form can be determined by using (2.3) or (3.9) for the standard power-law distribution. As stated earlier that (2.3) does not fully follow the inverse power law for the Karachi KAs land use density therefore D_a of KKAa cannot be fully assessed. Spectral analysis based on (2.3) in helpful to calculate the fractal dimension values. Thus, the wave number and the spectral density computation results as

$$W(n) = 988\, x^{-2.566} \qquad (3.11)$$

The goodness of fit is nearly $R^2 = 0.9997$, and B = 2.566, thus the estimated radial dimension value is attained as

$$D_a^* \approx \frac{2.566}{2} + 1 \approx 2.283 \qquad (3.12)$$

And the estimated profile dimension is yield.

$$D_s^* \approx \frac{5-2.566}{2} \approx 1.217 \qquad (3.13)$$

These results show that the fractal dimension can be explored either by the wave spectrum relation grounded on (2.3) or by the integral result of (2.3).

In fractal point of view, the land use forms and population of KKAs can be explored as: Firstly, the population density of KKAs obeys the general Clark's model, therefore the spatial dissemination (distribution) of the city (urban) population gives no self-similarity fractal property. Secondly, the dynamic process of land use and population bears self-affine fractal properties (Smeed, 1963; Chen, Zhou, 2008). Thirdly, the land use of these KAs of Karachi city takes on self-similar fractal topographies, meanwhile the fractal organization reprobates in a certain degree. Brownian motion may support the population pattern, but fBm is mostly responsible for the land use patterns (Smeed, 1963). The D_s of the population dissemination is close $D_s = 1{,}5$, that proposes that the H is near to o.5. So, the A_c of the spatial expansion arrangement is near to zero, and this esteem rehashes us of spatial region (Chen, 2008). The D_s values of land use and the corresponding H values are shown in **Table 1**. Thus, the A_c is estimated using these values that agrees a long memory and anti-permanence of spatial correlation between the periphery and urban core. The D_a is a size of self-similar form which gives the dimension of spatial distribution whereas, the D_s is one of the estimations of self-affine designs that represents the measurement of surface or a bend (curve) (Takayasu, 1990).

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Table 1. The mathematical (numerical) values of different fractal dimensions, scaling exponents, and autocorrelation coefficients.

Radial Dimension (D_a)	Profile Dimension (D_s)	Hurst Exponent (H)	Autocorrelation Coefficient (A_c)	Spectral Exponent (B)	Indirectly FD (D_s^*)	Directly Fractal Dimension (D)	Goodness of fit (R^2)
1.110	1.374	(0.626)	0.191	0.22	1.350	1.374	0.998
1.258	1.342	(0.658)	0.245	0.516	1.295	1.342	0.998
1.384	1.396	(0.604)	0.155	0.768	1.413	1.396	0.998
1.299	1.367	(0.633)	0.203	0.598	1.343	1.367	0.998
1.344	1.458	(0.542)	0.060	0.688	1.419	1.458	0.999
1.312	1.441	(0.559)	0.085	0.624	1.375	1.441	0.999

Table 2. Fractal dimension of district wise KKAs population and the related parametric values of 300 KAs.

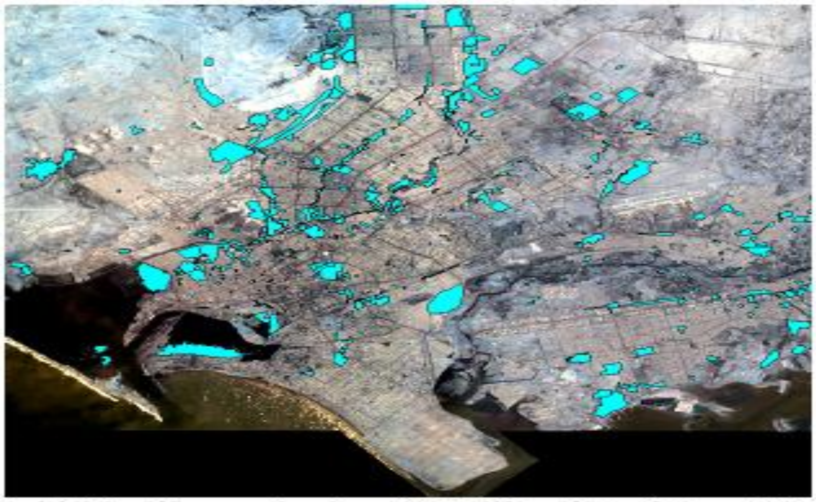
Sample Points	Frequency Sample Points	Cumulative Frequency	Cumulative Percent	FD of Population (H_P)	FD of Area (H_A)	FD of Density (H_D)	Total Population (PP)
CD	97	97	32	1.374 (0.626)	1.110 (0.89)	1.350 (0.650)	344667 (25.59)
ED	73	170	57	1.342 (0.658)	1.258 (0.715)	1.295 (0.705)	493742 (36.66)
WD	39	209	70	1.396 (0.604)	1.384 (0.616)	1.413 (0.587)	146325 (10.86)
MD	31	240	80	1.367 (0.633)	1.299 (0.701)	1.343 (0.657)	74405 (5.52)
SD	30	270	90	1.458 (0.542)	1.344 (0.656)	1.419 (0.581)	127194 (9.44)
KD	30	300	100	1.441 (0.559)	1.312 (0.688)	1.375 (0.625)	160601 (11.93)
Six Districts	300 KAs	-	-	-	-	-	1346935 (100 %)

The values in parentheses are the Hurst Exponent Values (H) for population(H_P), area (H_A), density (H_D) respectively and in right most column (PP) population percent in every district.

The areal expansion of Karachi with respect to its population is very divers and uncertain. Due to the rapid increase in KAs the peripheral area has turned into urbanized area in rapid time span which is shown in **Figure 1 & 2**. The fractal parameters of these 300 KAs and their existence (locality) according to the corresponding city districts have been displayed in **Table 2** for the further evidence and analysis.

Core-peripheral relationships of urban form of Karachi KAs are exemplified by the help of two fractal dimensions, i.e. radial dimension (D_a) and the profile dimension (D_s) which is shown in **Figure 3**. Further the behavior and the numerical results of directly fractal dimension (D) and indirectly fractal dimension (D_s^*) of KKAs is displayed in **figure 4** and additionally a comparative analysis of all four fractal parameters are examined in **figure 5** and **Table 1** which demonstrates a close numerical values and relation among these parameters respectively. **Figure 1** exhibits the overall overview of KKAs and their results and numerical results attained by this research have been shown in **table 2** and further demonstrated in **figures 6(a)** and **6(b).**

(**Figure 1**)



Katchi Abadis marked on Satellite data (Year 2005)

Source: National Space Agency of Pakistan

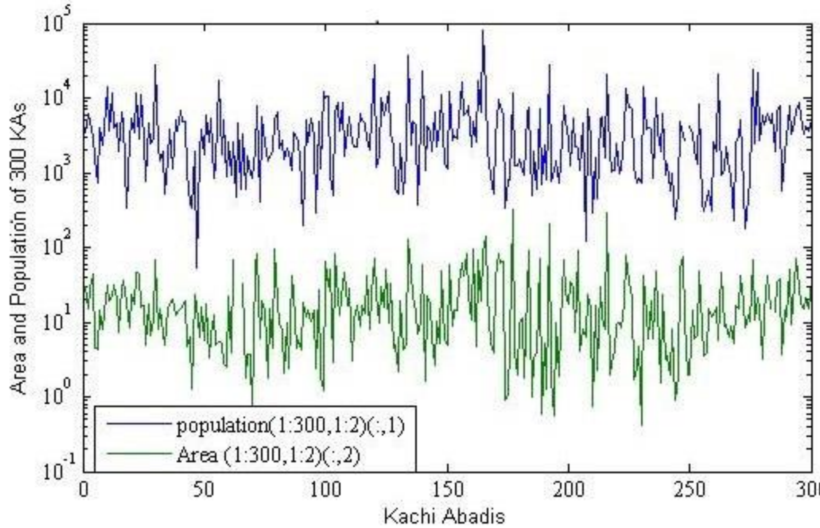


Figure 2 Areal expansion of Karachi w. r. t Population.

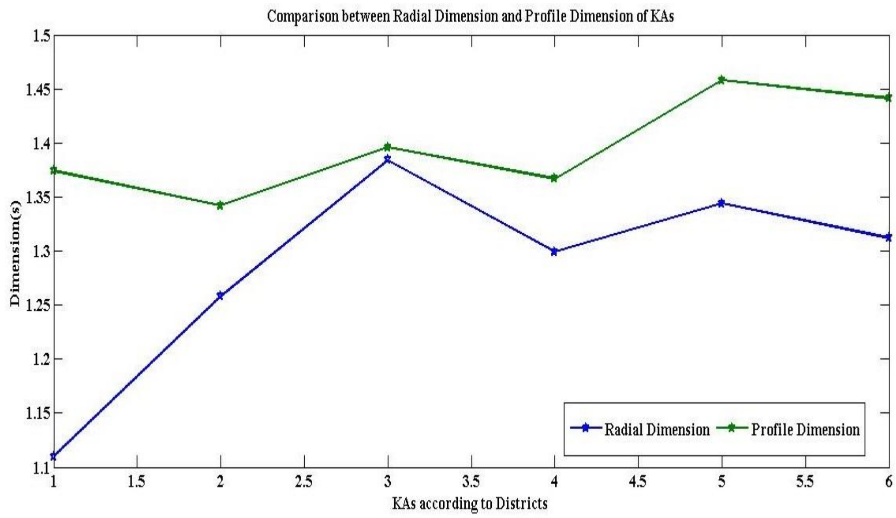


Figure 3 Comparison between (Radial Dimension) D_a and (Profile Dimension) D_s.

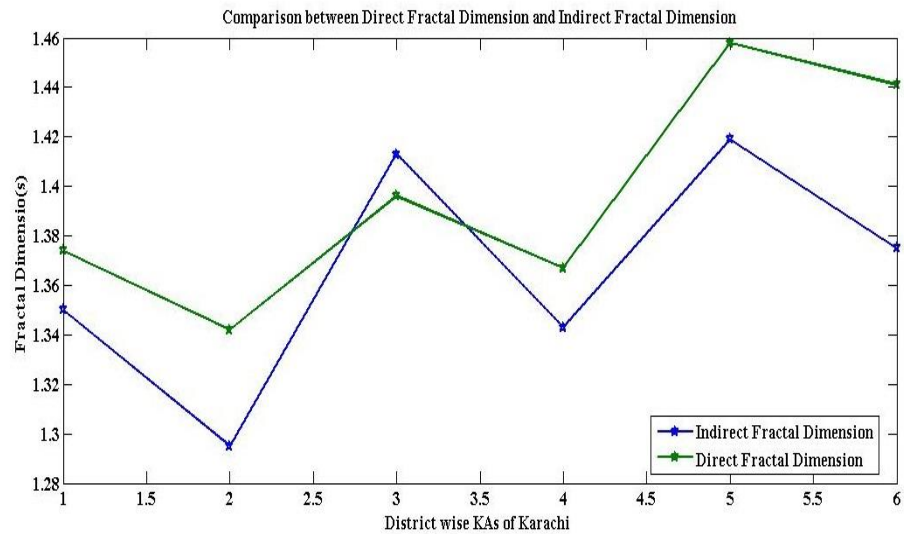


Figure 4 A comparison between Direct Fractal Dimension and Indirect Fractal Dimension.

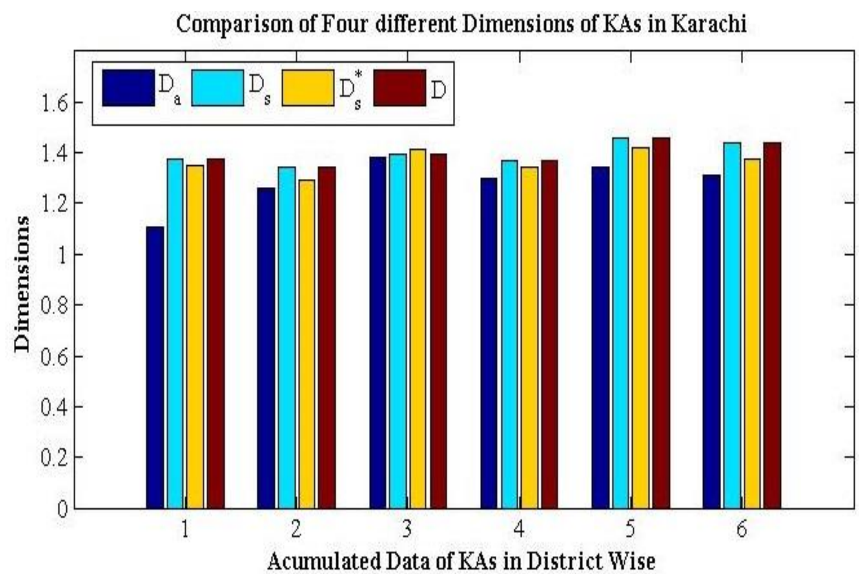
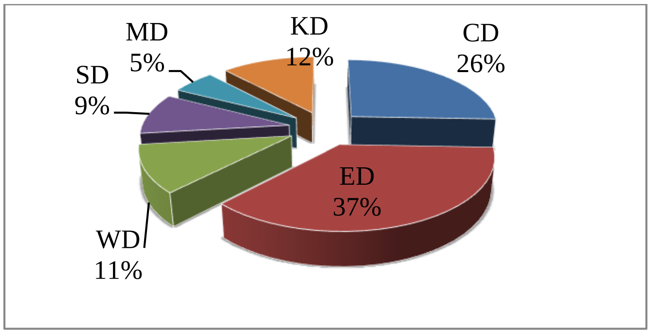


Figure 5 Comparison among all fractal parameters (dimensions)



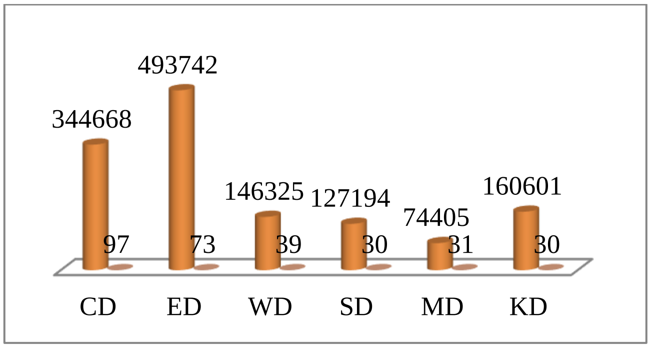


Fig. 6(a) District wise population of 300 KKAs, distribution and **Fig. 6(b)** population in percent.

Conclusion

A city behaves like a living organism. As it changes its physical structure and dimensions with the passage of time it possesses an open and dynamic system. The fractal investigation became a portion of urban morphology, grounded in scientific hypothesis, and created autonomously of urban morphology. There are several tools to study the fractal form of cities. Spectral analysis based on Fourier transformation is one of the most influential tools for the analysis of fractal cities. Initially, it can be assisted to disclose some important theoretical relations particularly between and. Further, it can be utilized to approximate fractal dimension indirectly and directly by the area-radius scaling methods as well. Mostly urban density obeys the inverse power law (IPL) but when this fails to follow properly spectral analysis is a requisite method of determining latent (hidden) fractal dimensions. Core-peripheral relationships of the urban form of the city's KAs are illustrated by the help of two fractal dimensions, i.e., the and the. Either the area-radius scaling is used to calculate the directly or indirectly the wave-spectrum relation is assessed. But this is mostly assessed with the wave-spectrum relations. Urban morphology does not have a characteristic scale, but the fractal dimension of urban morphology has a characteristic scale as fractal geometry possesses the self-similarity property means it has identical units of non-regularity on all scales. Various fractal parameters, such as D_a, D_s, D, *B*, and *H*, are numerically (mathematically) related to each other. However, the rational (judicious) range of these parameter values does not exactly match. The negative exponential model is transformed into a useful relation between density-density correlation and Hurst exponent by means of spectral analysis and Fourier transformation. This relation helped us to calculate the fractal dimension and density-density correlation functions of district-wise Kachi Abadis of Karachi.

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